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Nondiagonal Seed Universes and a Network of Double Gravitational Soliton Universes

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By using the double Ehlers transformations and γ transformations, new nondiagonal seed solutions are obtained. From these seed solutions we obtain a network of double gravitational soliton solutions. The double gravitational inverse scattering method is used to give some concrete examples of new solutions.

1. INTRODUCTION AND PREPARATION

In gravitational theory, the inverse scattering method found by Belinsky and Zakharov (BZ) (1978, 1979) has been developed into the double inverse scattering method by Zhong (1988a, b) and the latter can be connected with the double Ernst equation (Zhong, 1985; Ernst, 1968). By using the double inverse scattering method, we can easily give gravitational soliton solutions. However, in the process of using the inverse scattering method it is difficult to choose the seed solutions, especially the nondiagonal seed solutions. So far, there have been much fewer solutions of the double BZ equation used as the seed solutions. Gao and Zhong (1992) have discussed how to seek nondiagonal seed solutions. We have found some double Backlund transformations by which we can obtain a set of seed solutions. Furthermore, we can obtain a network of double soliton solutions.

First we introduce some necessary symbols and terms; let J denote the double imaginary unit, i.e., $J=i$ $(i^2=-1)$ or $J=\varepsilon$ $(\varepsilon^2=1,\ \varepsilon\neq\pm1)$. Let a be a real number set, $a=\{a_0, a_1, \ldots, a_n\}$, and $\sum_{n=1}^{\infty}|a_n|$ is a covergent series; then

$$a(J)=\sum_{n=0}^{\infty} a_n J^{2n} \tag{1.1}$$

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is called a double real number and corresponds to a dual real number pair (a_C, a_H),

$$\begin{aligned} a_C &= a(J=i) = \sum_{n=0}^{\infty} (-1)^n a_n \\ a_H &= a(J=\varepsilon) = \sum_{n=0}^{\infty} a_n \end{aligned} \tag{1.2}$$

If $a(J)$ and $b(J)$ are both double real numbers, then

$$z(J) = a(J) + J \cdot b(J) \tag{1.3}$$

is a double complex number. The double exponential function $e^{J\theta}$ is defined by

$$e^{J\theta} = C(J\theta) + JS(J\theta) \tag{1.4}$$

where

$$\begin{aligned} S(J\theta) &= \sum_{n=0}^{\infty} \frac{\theta}{(2n+1)!} (J\theta)^{2n} \\ C(J\theta) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} (J\theta)^{2n} \end{aligned} \tag{1.5}$$

where θ is a real number; when $J=i$, $S(J)=\sin$ and $C(J)=\cos$; when $j=\varepsilon$, $S(J)=\mathrm{sh}$ and $C(J)=\mathrm{ch}$. The commutation operator $\circ$ is defined as

$$\circ: \quad J \to \mathring{J}, \quad \mathring{i} = \varepsilon, \quad \mathring{\varepsilon} = i \tag{1.6}$$

The line element of an axisymmetric vacuum field (ASVF) can be taken as

$$ds^2 = f^{-1}[e^{\tau}(d\rho^2 + dz^2) + \rho^2\, d\phi^2] + f(dt + \omega\, d\phi)^2 \tag{1.7}$$

where f, ω, and τ are real functions of ρ and z only, and τ is determined by f and ω. Considering the double complex Ernst equation

$$\mathrm{Re}\, \mathscr{E}(J) \cdot \nabla^2 \mathscr{E}(J) = \nabla \mathscr{E}(J) \cdot \nabla \mathscr{E}(J) \tag{1.8}$$

where $F(J)$ and $\Omega(J)$ are double real functions of ρ and Z, if $\mathscr{E}(J)$ is a solution, then we obtain a pair of gravitational dual solutions,

$$\begin{cases} f = F_C, \\ \omega = V_{F_C}(\Omega_C), \end{cases} \qquad \begin{cases} \hat{f} = T(F_H) \\ \hat{\omega} = \Omega_H \end{cases} \tag{1.9}$$

where the NK transformations are defined by

$$\begin{aligned} T: \quad & F(J) \to T[F(J)] = \rho F^{-1}(J) \\ V: \quad & \Omega(J) \to V_F[\Omega(J)] = \omega \\ & \omega = \int \rho F^{-2}(J)[\partial_z \Omega(J) \cdot d\rho - \partial_\rho \Omega(J) \cdot dz] \end{aligned} \tag{1.10}$$

Let $M(J)$ be a 2×2 double matrix

$$M(J) = \frac{1}{F(J)} \begin{pmatrix} 1 & \Omega(J) \\ \Omega(J) & \Omega^2(J) - J^2 F(J) \end{pmatrix} \tag{1.11}$$

Then equation (1.8) is changed into

$$\begin{gathered} \partial_\rho[\rho \partial_\rho M(J) \cdot M^{-1}(J)] + \partial_z[\rho \partial_z M(J) \cdot M^{-1}(J)] = 0 \\ \det M(J) = -J^2, \qquad M^T(J) = M(J) \end{gathered} \tag{1.12}$$

where T denotes the transposition. From the solution $M(J)$ of equation (1.12) we can obtain the solution of equation (1.8),

$$\mathscr{E}(J) = 1/[M(J)]_{11} + J \cdot [M(J)]_{12}/[M(J)]_{11} \tag{1.13}$$

We consider the following double Lax pair:

$$\begin{aligned} \left(\partial_\rho + \frac{2\lambda\rho}{\rho^2 + \lambda^2} \partial_\lambda\right) \psi_0(J) &= \frac{\rho U_0(J) + \lambda W_0(J)}{\rho^2 + \lambda^2} \psi_0(J) \\ \left(\partial_z - \frac{2\lambda}{\rho^2 + \lambda^2} \partial_\lambda\right) \psi_0(J) &= \frac{\rho W_0(J) - \lambda U_0(J)}{\rho^2 + \lambda^2} \psi_0(J) \end{aligned} \tag{1.14}$$

$$\psi_0(\lambda = 0; J) = M_0(J) \tag{1.15}$$

where

$$U_0(J) = \rho \partial_\rho M_0(J) \cdot M_0^{-1}(J), \qquad W_0(J) = \rho \partial_z M_0(J) \cdot M^{-1}(J) \tag{1.16}$$

and $\psi_0(J) = \psi_0(\lambda, \rho, z; J)$ is a double ordinary complex matrix, λ is a double ordinary complex parameter, and the n-soliton solutions are obtained as

$$
\begin{aligned}
M_n(J) &= |\det M'_n(J)|^{-1/2}\, M'_n(J) \\
[M'_n(J)]_{ab} &= [M_0(J)]_{ab} - \sum_{k,\,l=1}^{n} N_a^{(k)}(J)[\Gamma^{-1}(J)]_{kl}\, N_b^{(k)}(J)/\mu_l(J)\,\mu_k(J) \\
[\Gamma(J)]_{kl} &= m_a^{(k)}(J)[M_0(J)]_{ab}\, m_b^{(l)}(J)/[\mu_k(J)\,\mu_l(J)+\rho^2] \\
N_a^{(k)}(J) &= m_b^{(k)}(J)[M_0(J)]_{ab} \\
m_a^{(k)}(J) &= m_{0b}^{(k)}[\psi_0^{-1}(\lambda=\mu_k(J); J]_{ab} \\
\det M'_n(J) &= -J^2(-1)^n\,\rho^{2n}\prod_{k=1}^{n}\mu_k^{-2}(J)
\end{aligned}
\tag{1.17}
$$

where $m_{0b}^{(k)}(J)$ are double constants (a, $b=1$, 2) and

$$
\begin{aligned}
\partial_\rho U_k(J) &= 2\rho\mu_k(J)/[\rho^2+\mu_k^2(J)] \\
\partial_z \mu_k(J) &= -2\mu_k^2(J)/[\rho^2+\mu_k^2(J)] \\
\mu_k(J) = \mu_k(\rho, z; J) &= \alpha_k(J) - z \pm \{[\alpha_k(J)-z]^2+\rho^2\}^{1/2}
\end{aligned}
\tag{1.18}
$$

where $\alpha_k(J)$ are constants. Let

$$
\bar{M}_n = \begin{cases} M_n(J) & \text{when } n \text{ is even} \\ M_n(\mathring{J}) & \text{when } n \text{ is odd} \end{cases}
\tag{1.19}
$$

Then we obtain

$$
\mathscr{E}_n(J) = 1/[\bar{M}_n(J)]_{11} + J\cdot[\bar{M}_n(J)]_{12}/[\bar{M}_n(J)]_{11}
\tag{1.20}
$$

In the system (1.17), $\psi_0(J)$ only enters along the pole's trajectories $\mu_k(J)$ (Letelier, 1985). In order to construct the soliton solutions we only need $\psi_{0k}(J)=\psi_0(\lambda=\mu_k(J), \rho, z; J)$, $(k=1, 2, \ldots)$; the condition (1.15) reads

$$
\psi_{0k}(J)\,|_{\mu_k(J)\to 0} = M_0(J)
\tag{1.21}
$$

Notice that (Letelier, 1985; Gao and Zhong, 1992)

$$
\begin{aligned}
&\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2\right)\ln \mu_k(J) = 0 \\
&\left.\frac{\partial_\rho \mu_k(J)}{2\mu_k(J)}\right|_{\mu_k(J)\to 0} = \frac{1}{\rho}, \qquad \left.\frac{\partial_z \mu_k(J)}{2\mu_k(J)}\right|_{\mu_k(J)\to 0} = 0
\end{aligned}
\tag{1.22}
$$

Considering the function $\varphi(\rho, z)$ which satisfies $\nabla^2\varphi(\rho, z) = 0$, and the operator $\nabla^2 = \partial_\rho^2 + (1/\rho)\partial_\rho + \partial_z^2$, we have

$$Y_k(J) = \int \tfrac{1}{2}[\rho/\mu_k(J)]\{[\partial_\rho \mu_k(J) \cdot \partial_\rho \varphi - \partial_z \mu_k(J) \cdot \partial_z \varphi]\, d\rho + [\partial_z \mu_k(J) \cdot \partial_\rho \varphi - \partial_\rho \mu_k(J) \cdot \partial_z \varphi]\, dz\} \tag{1.23}$$

When $\mu_k \to 0$, $Y_k[\varphi, \mu_k(J)] = \varphi$.

According to Gao and Zhong (1992), if the solution $M_0(J)$ of (1.12) satisfies (i)

$$M_0(J) = M_0(\varphi_1, \varphi_2, \ldots, \varphi_N; J) \tag{1.24a}$$

and (ii)

$$\frac{\partial}{\partial \varphi_i}\left\{\frac{\partial}{\partial \varphi_j} M_0[\varphi_1, \varphi_2, \ldots, \varphi_N; J] \cdot M_0^{-1}[\varphi_1, \varphi_2, \ldots, \varphi_N; J]\right\} = 0 \tag{1.24b}$$

then we can obtain directly a wave function $\psi_{0k}(J)$

$$\psi_{0k}(J) = M_0\{\varphi_1 \to Y_k[\varphi_1, \mu_k(J)] \cdots \varphi_N \to Y_k[\varphi_N, \mu_k(J)]; J\} \tag{1.25}$$

where the arrow denotes that φ_N is replaced by $Y_k(J)$. In particular, when the seed solution $M_0(J) = M_0(\varphi; J)$, the condition (ii) is satisfied automatically.

2. GENERATING OF NONDIAGONAL SEED SOLUTION AND THE NETWORK OF DOUBLE SOLITON SOLUTIONS

In Section 1 we introduced the two conditions for the seed solutions. It is still difficult to find more new seed solutions. In order to solve this problem we introduce some nonlinear transformations, by use of which we can obtain new seed solutions. We have found that two transformations satisfy the requirement, the double Ehlers transformation and the double γ transformation (Zhong, 1988a, b).

Let $\mathscr{E}(J) = F_0(J) + J \cdot \Omega_0(J)$ be a double solution of equation (1.8); by the double γ transformation

$$\begin{aligned} T\gamma\colon\quad & \mathscr{E}(J) \to \mathscr{E}'(J) = F_0'(J) + J \cdot \Omega_0'(J) \\ & F_0'(J) = -J^2\gamma^{-2}F_0(J)/[\Omega_0^2(J) - J^2F_0(J)] \\ & \Omega_0'(J) = J^2\gamma^{-2}\Omega_0(J)/[\Omega_0^2 - J^2F_0^2(J)] \end{aligned} \tag{2.1}$$

For equation (1.12) we have

$$T\gamma:\quad M_0(J)\to M_0'(J)$$

$$M_0(J)=\frac{1}{F_0(J)}\begin{pmatrix}1 & \Omega_0(J)\\ \Omega_0(J) & \Omega_0^2(J)-J^2F_0(J)\end{pmatrix}\tag{2.2}$$

$$M_0'(J)=\frac{1}{F_0(J)}\begin{pmatrix}\gamma^2[\Omega_0^2(J)-J^2F_0^2(J)] & -J^2\Omega_0(J)\\ -J^2\Omega_0(J) & \gamma^2\end{pmatrix}$$

Clearly the solution $M_0'(J)$ of equation (1.12) satisfies equations (1.24a) and (1.24b) if $M_0(J)$ satisfies them; i.e., if $M_0(J)$ is a seed solution, then $M_0'(J)$ is also a seed solution. The Ehlers transformation acting on the original seed solution $M_0(J)$ yields a different situation, and we will discuss this in the following.

Let $\mathscr{E}(J)=F_0(J)+J\Omega_0(J)$ be an original seed solution, and let the Ehlers transformation act on it,

$$\begin{aligned}
&T_E:\quad \mathscr{E}_0(J)\to\mathscr{E}_0'(J)=F_0'(J)+J\Omega_0'(J)\\
&\qquad\mathscr{E}(J)=[a(J)\,\mathscr{E}_0(J)+Jb(J)]/[Jc(J)\,\mathscr{E}_0(J)+d(J)]\\
&a(J)\,d(J)-J^2b(J)\,c(J)=1\\
&T_E:\quad M_0(J)\to M_0'(J)\\
&M_0'(J)=\begin{pmatrix}\Delta_0(J)/\Delta_1(J) & \Delta_2(J)/\Delta_1(J)\\ \Delta_2(J)/\Delta_1(J) & [\Delta_2^2(J)-J^2\Delta_1^2(J)]/\Delta_0(J)\,\Delta_1(J)]\end{pmatrix}
\end{aligned}\tag{2.3}$$

where

$$\begin{aligned}
F_0'(J)&=\Delta_1(J)/\Delta_0(J)\\
\Omega_0'(J)&=\Delta_2(J)/\Delta_0(J)\\
\Delta_0(J)&=[d^2(J)+J^2c(J)\,\Omega_0(J)]-J^2c(J)\,F_0(J)\\
\Delta_1(J)&=F_0(J)\\
\Delta_2(J)&=J^2a(J)\,c(J)\,\Omega_0^2(J)+[a(J)\,d(J)+J^2b(J)\,c(J)]\,\Omega_0(J)\\
&\quad-a(J)\,c(J)\,F_0^2(J)+b(J)\,d(J)
\end{aligned}\tag{2.4}$$

From (2.3) and (2.4) we know that if $M_0(J)$ satisfies the condition (1.24a), then $M_0'(J)$ satisfies it. In the case of $M_0(J)=M_0(\varphi;J)$, $M_0'(J)=M_0'(\varphi;J)$ satisfies automatically (1.24b), i.e., if $M_0(\varphi;J)$ is a seed solution, then $M_0'(\varphi;J)$ is also a seed solution.

Since the result of using successively the Ehlers transformation is still an Ehlers transformation, and the result of using successively a γ transformation two times is equal to the identical transformation, we should use alternately the Ehlers transformation and the γ transformation. The set of seed solutions and the network of new soliton solutions are obtained as follows:

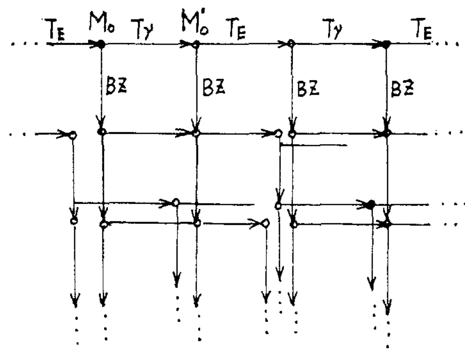

Fig. 1.

Notice that the γ transformation and the BZ transformation are commutive (Zhong, 1990), but the Ehlers transformation and the BZ transformation are not.

3. NEW DOUBLE GRAVITATIONAL SOLITON SOLUTIONS FOR THE ASVF CASE

For the sake of convenience, we write the seed solution as

$$M_0(J)=\begin{pmatrix}\Delta_0(J)/\Delta_1(J) & \Delta_2(J)/\Delta_1(J)\\ \Delta_2(J)/\Delta_1(J) & [\Delta_2^2(J)-J^2\Delta_1^2(J)]/\Delta_0(J)\,\Delta_1(J)\end{pmatrix}$$
$$\det M_0(J)=-J^2,\qquad M_0^T(J)=M_0(J) \tag{3.1}$$

The corresponding double scattering wave function is

$$\psi_{0k}=M_0\{\varphi_1\to Y_k[\varphi_1;J]\cdots\varphi_N\to Y_k[\varphi_N;J];J\} \tag{3.2}$$

The one-soliton solutions $M_1(J)$ associated with seed solutions are

$$M_1(J)=\frac{1}{[M_1']_{11}\,[M_1']_{22}-[M_1']_{12}\,[M_1']_{21}}\begin{pmatrix}[M_1']_{11} & [M_1']_{12}\\ [M_1']_{21} & [M_1']_{22}\end{pmatrix} \tag{3.3}$$

$$[M_1'(J)]_{11} = [M_0(J)]_{11} - \{[\mu_1^2(J) + \rho^2]/B\}(\{k_1[M_0(J)]_{11} + k_2[M_0(J)]_{12}\}[M_0(J)]_{11} + \{k_2[M_0(J)]_{11} + k_3[M_0(J)]_{21}\}[M_0(J)]_{12})$$

$$[M_1'(J)]_{12} = [M_0(J)]_{12} - \{[\mu_1^2(J) + \rho^2]/B\}(\{k_1[M_0(J)]_{11} + k_2[M_0(J)]_{21}\}[M_0(J)]_{12} + \{k_2[M_0(J)]_{12} + k_3[M_0(J)]_{21}\}[M_0(J)]_{22})$$

$$[M_1'(J)]_{22} = [M_0(J)]_{22} - \{[\mu_1^2(J) + \rho^2]/B\}(\{k_1[M_0(J)]_{12} + k_2[M_0(J)]_{22}\}[M_0(J)]_{12} + \{k_2[M_0(J)]_{12} + k_3[M_0(J)]_{22}\}[M_0(J)]_{22})$$

$$[M_1'(J)]_{21} = [M_1'(J)]_{12} \tag{3.4}$$

$$k_1 = [m_1^{(1)}(J)]^2, \qquad k_2 = [m_1^{(1)}(J)][m_2^{(1)}(J)], \qquad k_3 = [m_2^{(1)}(J)]^2$$

$$B = \{k_1[M_0(J)]_{11} + 2k_2[M_0(J)]_{12} + k_3[M_0(J)]_{22}\}\,\mu_1^2(J) \tag{3.5}$$

$$m_1^{(1)}(J) = -J^2\left[m_1^{0(1)}(J)\frac{\Delta_2^2(Y_k;J) - J^2\Delta_1^2(Y_k;J)}{\Delta_0(Y_k;J)\,\Delta_1(Y_k;J)} - m_2^{0(1)}(J)\frac{\Delta_2(Y_k;J)}{\Delta_1(Y_k;J)}\right]$$

$$m_2^{(1)}(J) = -J^2\left[-m_1^{0(1)}(J)\frac{\Delta_2(Y_k;J)}{\Delta_1(Y_k;J)} + m_2^{0(1)}(J)\frac{\Delta_0(Y_k;J)}{\Delta_1(Y_k;J)}\right] \tag{3.6}$$

3.1. We take the double Weyl-type solution as the original seed solution. After taking the double Ehlers transformation, we obtain the new double seed solution

$$M_0(J) = \begin{pmatrix} \Delta_0(\varphi)/\Delta_1(\varphi) & \Delta_2(\varphi)/\Delta_1(\varphi) \\ \Delta_2(\varphi)/\Delta_1(\varphi) & [\Delta_2^2(\varphi) - J^2\Delta_1^2(\varphi)]/\Delta_0(\varphi)\cdot\Delta_1(\varphi) \end{pmatrix}$$

$$\det M_0(J) = -J^2, \qquad \nabla^2\varphi(\rho, z) = 0$$

$$\Delta_0(\varphi) = d^2(J) - J^2c^2(J)e^{2\varphi}$$

$$\Delta_1(\varphi) = e^{\varphi}$$

$$\Delta_2(\varphi) = b(J)\,d(J) - a(J)\,c(J)e^{2\varphi}$$

$$a(J)\,d(J) - J^2b(J)\,c(J) = 1 \tag{3.7}$$

$$\psi_{0k}(J) = \begin{pmatrix} \Delta_0(Y_k)/\Delta_1(Y_k) & \Delta_2(Y_k)/\Delta_1(Y_k) \\ \Delta_2(Y_k)/\Delta_1(Y_k) & [\Delta_2^2(Y_k) - J^2\Delta_1^2(Y_k)]/\Delta_0(Y_k)\,\Delta_1(Y_k) \end{pmatrix} \tag{3.8}$$

From (3.3)–(3.5) we know that if we compute $m_1^{(1)}(J)$ and $m_2^{(1)}(J)$ we obtain the one-soliton solution

$$\begin{aligned} m_1^{(1)}(J) &= -J^2(L_1 e^{4Y_1} + L_2 e^{2Y_1} + L_3)/(\beta_1 e^{3Y_1} + \beta_2 e^{Y_1}) \\ m_2^{(1)}(J) &= -J^2 e^{-Y_1}(S_1 e^{2Y_1} + S_2) \end{aligned} \tag{3.9}$$

where

$$L_1 = p_1\alpha_1^2 - q_1\alpha_1\alpha_2, \qquad L_2 = p_1(2\alpha_1\alpha_2 - J^2) - q_1(\alpha_1\beta_2 + \alpha_2\beta_1)$$

$$L_3 = p_1\alpha_2^2 - q_1\alpha_2\beta_2, \qquad S_1 = -p_1\alpha_1 + q_1\beta_1, \qquad S_2 = -p_1\alpha_2 + q_1\beta_2$$

$$\alpha_1 = -a(J)\,c(J), \qquad \alpha_2 = b(J)\,d(J), \qquad \beta_1 = -J^2c^2(J)$$

$$\beta_2 = d^2(J), \qquad p_1 = m_{01}^{(1)}(J), \qquad q_1 = m_{02}^{(1)}(J)$$

3.2. Considering the double Weyl-type solution as the original seed solution, and by taking the double Ehlers transformation and the double γ transformation in turn, we obtain the new double seed solution

$$M_0(J) = \begin{pmatrix} \Delta_0(\varphi)/\Delta_1(\varphi) & \Delta_2(\varphi)/\Delta_1(\varphi) \\ \Delta_2(\varphi)/\Delta_1(\varphi) & [\Delta_2^2(\varphi) - J^2\Delta_1^2(\varphi)]/\Delta_0(\varphi)\,\Delta_1(\varphi) \end{pmatrix}$$

$$\det M_0(J) = -J^2, \qquad \nabla^2\varphi(\rho, z) = 0$$

$$\begin{aligned} \Delta_0(\varphi) &= v_1 e^{4\varphi} + v_2 e^{2\varphi} + v_3 \\ \Delta_1(\varphi) &= u_1 e^{3\varphi} + u_2 e^{\varphi} \\ \Delta_2(\varphi) &= w_1 e^{4\varphi} + w_2 e^{2\varphi} + w_3 \end{aligned} \tag{3.10}$$

$$v_1 = \alpha_1^2, \qquad v_2 = 2\alpha_1\alpha_2 - J^2, \qquad v_3 = \alpha_2^2$$

$$u_1 = -J^2\gamma^{-2}\beta_1, \qquad u_2 = -J^2\gamma^{-2}\beta_2$$

$$w_1 = J^2\gamma^2\alpha_1\beta_1, \qquad w_2 = J^2\gamma^{-2}(\alpha_1\beta_2 + \alpha_2\beta_1), \qquad w_3 = J^2\alpha_2\beta_2$$

In this case, $m_1^{(1)}(J)$ and $m_2^{(1)}(J)$ are given by

$$m_1^{(1)}(J) = -J^2\,\frac{L_1^{(1)}e^{8Y_1} + L_2^{(1)}e^{6Y_1} + L_3^{(1)}e^{4Y_1} + L_4^{(1)}e^{2Y_1} + L_5^{(1)}}{L_1^{(2)}e^{7Y_1} + L_2^{(2)}e^{5Y_1} + L_3^{(2)}e^{3Y_1} + L_4^{(2)}e^{Y_1}}$$

$$m_2^{(1)}(J) = \frac{S_1 e^{4Y_1} + S_2 e^{2Y_1} + S_3}{u_1 e^{3Y_1} + u_2 e^{Y_1}} \tag{3.11}$$

$$L_1^{(1)}=p_1 w_1^2-q_1 v_1 w_1$$
$$L_2^{(1)}=p_1(2w_1 w_2-u_1^2)-q_1(v_1 w_2+v_2 w_1)$$
$$L_3^{(1)}=p_1(2w_1 w_3+w_2^2-2u_1 u_2)-q_1(v_1 w_3+v_2 w_2+v_3 w_1)$$
$$L_4^{(1)}=p_1(2w_3 w_2-u_2^2)-q_1(v_2 w_3+v_3 w_2)$$
$$L_5^{(1)}=p_1 w_3^2-q_1 v_3 w_2$$
$$L_1^{(2)}=v_1 u_1, \qquad L_2^{(2)}=v_1 u_2+v_2 u_1, \qquad L_3^{(2)}=v_3 u_1, \qquad L_4^{(2)}=v_3 u_2$$
$$S_1=-p_1 w_1, \qquad S_2=-p_1 w_2+q_1 v_2, \qquad S_3=-p_1 w_3+q_1 v_3$$
$$p_1=m_1^{0(1)}(J), \qquad q_1=m_2^{0(1)}(J)$$

This is the explicit expression for the one-soliton solution associated with the seed solution $M_0(J)$.

3.3. Similarly, considering the solution of the hyperbolic complex Ernst equation as the original seed solution,

$$M_{0H}=\begin{pmatrix}\varphi & 1\\ 1 & 0\end{pmatrix}$$
$$\det M_{0H}=-1, \qquad \nabla^2\varphi(\rho, z)=0 \tag{3.12}$$

and taking the Ehlers transformation on the hyperbolic complex seed solution M_{0H}, we obtain the new seed solution

$$R_{0H}=\begin{pmatrix}\Delta_0(\varphi)/\Delta_1(\varphi) & \Delta_2(\varphi)/\Delta_1(\varphi)\\ \Delta_2(\varphi)/\Delta_1(\varphi) & [\Delta_2^2(\varphi)-J^2\Delta_1^2(\varphi)]/\Delta_0(\varphi)\,\Delta_1(\varphi)\end{pmatrix}$$
$$\det R_{0H}=-1, \qquad \nabla^2\varphi(\rho, z)=0 \tag{3.13}$$
$$\Delta_0(\varphi)=\beta_1\varphi^2+\beta_2\varphi, \qquad \Delta_1(\varphi)=\varphi, \qquad \Delta_2(\varphi)=\theta_1\varphi^2+\theta_2\varphi$$
$$\beta_1=d^2, \qquad \beta_2=2dc, \qquad \theta_1=bd, \qquad \theta_2=ad+bc, \qquad ad-bc=1$$

where a, b, c, and d are real constants.

In this case, we have $m_1^{(1)}$ and $m_2^{(1)}$ as

$$m_1^{(1)}=-[L_1 Y_{1H}^3+L_2 Y_{1H}+L_3 Y_{1H}]/[\beta_1 Y_{1H}+\beta_2 Y_{1H}]$$
$$m_2^{(1)}=S_0+S_1 Y_{1H}$$
$$L_1=p_{1H}(\theta_2^2-1)-q_{1H}\beta_2\theta_2, \qquad L_2=2p_{1H}\theta_1\theta_2-q_{1H}(\beta_1\theta_2+\beta_2\theta_1) \tag{3.14}$$
$$L_3=p_{1H}\theta_1^2-q_{1H}\beta_1\theta_1$$
$$S_1=-p_{1H}\theta_1+q_{1H}\beta_1, \qquad S_0=-p_{1H}\theta_2+q_{1H}\beta_2$$

where p_{1H} and q_{1H} are constants.

4. DISCUSSION FOR THE CASE OF CSVF

For the case of a cylindrically symmetric vacuum field (CSVF), the line element can be written as

$$ds^2 = g^{-1}[e^{2\delta}(d\rho^2 - dt^2) + \rho^2\, d\phi^2] + g(dz + \sigma\, d\phi)^2 \tag{4.1}$$

where g, δ, and σ are functions of ρ and t only, and σ is determined by g and δ; we obtain the double Ernst equation

$$\mathrm{Re}(\mathscr{E})\,\tilde{\nabla}^2\mathscr{E} = \tilde{\nabla}\mathscr{E}\cdot\tilde{\nabla}\mathscr{E} \tag{4.2}$$

with the operators $\tilde{\nabla}^2 = \partial^2\rho + (1/\rho)\partial_\rho - \partial_t^2$, $\tilde{\nabla} = (\partial_\rho, i\partial_t)$, and $\mathscr{E} = \mathscr{E}(\rho, t) = G(\rho, t) + i\Sigma(\rho, t)$ is an ordinary complex Ernst potential. A pair of dual CSVF solutions are

$$\begin{cases} g = G \\ \sigma = \tilde{V}_G(\Sigma) \end{cases} \qquad \begin{cases} \hat{g} = \hat{T}(G) \\ \hat{\sigma} = \Sigma \end{cases} \tag{4.3}$$

The NK transformation is defined as

$$\begin{aligned} &\tilde{T}\colon\quad G \to T(G) = \rho/G \\ &\tilde{V}\colon\quad \Sigma \to \tilde{V}(\Sigma) = \sigma \\ &\sigma = \int (\rho/G^2)(\partial_t\Sigma\cdot d\rho + \partial_\rho\Sigma\cdot dt) \end{aligned} \tag{4.4}$$

Let

$$\tilde{M} = \frac{1}{G}\begin{pmatrix} 1 & \Sigma \\ \Sigma & \Sigma^2 + G^2 \end{pmatrix}$$

We obtain the BZ equation for the case of CSVF (Zhong, 1990)

$$\begin{gathered} \partial_\rho(\partial_\rho M\cdot\tilde{M}^{-1}) - \partial_t(\rho\partial_t\tilde{M}\cdot\tilde{M}^{-1}) = 0 \\ \det\tilde{M} = 1, \qquad \tilde{M}^T = \tilde{M} \end{gathered} \tag{4.5}$$

From the solution of equation (4.5), we can obtain the solution of equation (4.2),

$$\mathscr{E} = 1/[\tilde{M}]_{11} + i[\tilde{M}]_{12}/[\tilde{M}]_{11} \tag{4.6}$$

Similar to the case of ASVF, we can easily write out the set of the new seed solutions of equation (4.5) and the network of the corresponding soliton solutions.

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